On computable operations

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The theory of algorithms uses the notions of a computable function, a decidable set, and an enumerable set (partial recursive function, recursive set, recursively enumerable set as defined in [2]). Recent research in the theory of algorithms started to use the notions of a function that is computable relative to some other functions (is computably reducible to those functions) and of a set that is decidable relative to some other sets (is (decidably) reducible to those sets). The exact definitions that formalize these intuitive notions were given by Kleene and Post [2, 3]: A function φ is computably reducible to functions $\psi_1, \dots, \psi_l$ if φ is partial recursive relative to $\psi_1, \dots, \psi_l$. A set P is (decidably) reducible to sets $Q_1, \dots, Q_l$ if its characteristic function is computably reducible to the characteristic functions of $Q_1, \dots, Q_l$.

In this note, we study the natural notion of a set that is enumerable relative to other sets (a set that is enumeration reducible [сводящегося по перечислимости] to other sets). This notion of enumeration reducibility allows us to give equivalent definitions of two other reducibilities mentioned above (Corollary to Theorem 7 and Theorem 8). The definition of *enumeration reducibility* is given in Section 7; it uses the notion of a computable operation that is introduced in Section 4. In Section 5, we establish a connection between computable operations and another definition suggested by A.N. Kolmogorov, and in Section 6, we establish the relation between computable operations and some constructions of Post (that appeared even earlier). Sections 1–3 are of an introductory nature.

1. Families of sets as topological spaces.

We consider an arbitrary family $\mathcal{T}$ of sets as a T_0-topological space with the following topology: For every finite set F, the subfamily $\mathcal{O}_F \subseteq \mathcal{T}$ of all supersets of F is an elementary open set; open sets are arbitrary unions of elementary open sets. In particular, for an arbitrary set M, the family $\mathcal{T}_M$ of all its subsets and the family $\mathcal{T}'_M$ of all its finite subsets become a connected compact [бикомпактное][1] T_0-space.

Lemma. *Let $M_1, \dots, M_l$ be arbitrary sets, and let $\mathcal{T}$ be an arbitrary family of sets. The mapping $f : \mathcal{T}_{M_1} \times \dots \times \mathcal{T}_{M_l} \to \mathcal{T}$ is continuous if and only if*

$$f(X_1, \dots, X_l) = \bigcup_{\substack{\{X'_1 \subseteq X_1, \dots, X'_l \subseteq X_l; \\ X'_1 \in \mathcal{T}'_{M_1}, \dots, X'_l \in \mathcal{T}'_{M_l}\}}} f(X'_1, \dots, X'_l).$$

for all $X_1 \subseteq M_1$, ..., $X_l \subseteq M_l$.

[1]The notion “bicompact” was used for compact spaces by Moscow topologists at the time.

2. The set $\mathcal{H}$.

The theory of algorithms has to consider, in addition to the set $\mathbb{N}$ of natural numbers, also the set $\mathbb{N}^2$ of ordered pairs of natural numbers; more generally, the set $\mathbb{N}^n$ of all n-tuples of natural numbers, the set $\mathbb{N}_2 = \bigcup_k \mathbb{N}^k$ of all tuples of natural numbers (of any finite length), the set $\mathbb{N}_3 = \bigcup_k \mathbb{N}_2^k$ of all tuples formed by elements of $\mathbb{N}_2$, etc. It is convenient (following [1]) to consider directly a set $\mathcal{H}$ of all "combinations of a symbol | with itself." The set $\mathcal{H}$ is defined [inductively] as the least set such that

a) the "base element" | and "empty element" Λ are in $\mathcal{H}$;
b) if $\mathrm{a} \in \mathcal{H}$, then $(\mathrm{a}) \in \mathcal{H}$; and
c) if $\mathrm{a} \in \mathcal{H}$ and $\mathrm{b} \in \mathcal{H}$, then $\mathrm{ab} \in \mathcal{H}$.

(Here, ab is a concatenation of a and b, and we let $\Lambda\mathrm{a} = \mathrm{a}\Lambda = \mathrm{a}$). We embed natural numbers in $\mathcal{H}$ by identifying 0 with Λ, 1 with |, 2 with ||, etc. An n-tuple $\mathrm{a}_1, \dots, \mathrm{a}_n$ of elements of $\mathcal{H}$ can then be identified with $(\mathrm{a}_1) \dots (\mathrm{a}_n) \in \mathcal{H}$. In this way, all sets $\mathbb{N}^k$ and $\mathbb{N}_k$ become enumerable subsets of $\mathcal{H}$ (see below). We denote the families $\mathcal{T}_{\mathcal{H}}$ of all subsets of $\mathcal{H}$ and $\mathcal{T}'_{\mathcal{H}}$ of all finite subsets of $\mathcal{H}$ by $\mathcal{V}$ and $\mathcal{V}'$, respectively. Every element $(\mathrm{a}_1) \dots (\mathrm{a}_n) \in \mathcal{H}$ is considered as a representative of a finite (unordered) subset $\{\mathrm{a}_1, \dots, \mathrm{a}_n\} \in \mathcal{V}'$; in this way, every element of $\mathcal{V}'$ of cardinality n has $n!$ representatives.

A bijective correspondence between a subset of $\mathbb{N}$ and $\mathcal{H}$ is called a (bijective) numbering of $\mathcal{H}$. A numbering is admissible if there exist computable functions $\alpha(m)$ and $\beta(m, n)$ that yield the numbers of (m) and mn from numbers m and n of m and n. We call a set $R \subseteq \mathcal{H}$ enumerable if the set of the corresponding numbers (in an admissible numbering) is enumerable. This definition can be extended to all classes P_n in the Kleene–Mostowski hierarchy [6] in a natural way. One can prove that the notion of enumerability (and of all classes P_n) do not depend on the choice of an admissible numbering.

3. Partial mappings.

A mapping of a set $E \subseteq X$ to Y is called a partial mapping from X to Y. If X and Y are topological spaces, we may consider partial continuous mappings from X to Y [using the induced topology on $E \subseteq X$]. The graph of a partial mapping ψ from $\mathcal{H}^l$ to $\mathcal{H}$ is the set G_ψ of all elements $(\mathrm{x}_1) \dots (\mathrm{x}_l)\mathrm{y} \in \mathcal{H}$ such that $\mathrm{y} = \psi(\mathrm{x}_1, \dots, \mathrm{x}_l)$. The graph of a partial mapping Ψ of $\mathcal{H}^l$ to $\mathcal{V}$ is the set G_Ψ of all elements $(\mathrm{x}_1) \dots (\mathrm{x}_l)(\mathrm{y}) \in \mathcal{H}$ such that $\mathrm{y} \in \Psi(\mathrm{x}_1, \dots, \mathrm{x}_l)$. A partial mapping [of these types] is called computable if its graph is an enumerable set. Every partial mapping F from $(\mathcal{V}')^l$ to $\mathcal{V}$ induces a partial mapping $\tilde{F}$ from $\mathcal{H}^l$ to $\mathcal{V}$ such that $\tilde{F}(\mathrm{x}_1, \dots, \mathrm{x}_l) = F(\xi_1, \dots, \xi_l)$, where $\mathrm{x}_i \in \mathcal{H}$ is a representative of a finite set $\xi_i \in \mathcal{V}'$ ($i = 1, \dots, l$); we say that F is computable when $\tilde{F}$ is.

4. Computable operations.

Let $M_1, \dots, M_l$ be enumerable subsets of $\mathcal{H}$. A [total] mapping of $\mathcal{T}_{M_1} \times \dots \times \mathcal{T}_{M_l}$ to $\mathcal{V}$ is an l-ary computable operation if (a) it is continuous and (b) the induced mapping $(\mathcal{V}')^l \to \mathcal{V}$ is computable. Using the Lemma from Section 1, we can prove the following results:

Theorem 1. *The composition of computable operations is a computable operation.*

Theorem 2. *Every computable continuous partial mapping from $(\mathcal{V}')^l$ to $\mathcal{V}$ can be extended to a computable operation.*

Theorem 3. *For every l-ary computable operation* U *and all sets* $E_1, \dots, E_l, D$ *that are enumerable subsets of* $\mathcal{H}$*, there exists a computable operation* U_1 *that is a mapping of* $\mathcal{T}_{E_1} \times \dots \times \mathcal{T}_{E_l}$ *to* $\mathcal{T}_D$ *such that* $\mathrm{U}(S_1, \dots, S_l) = R$ *implies* $\mathrm{U}_1(S_1, \dots, S_l) = R$ *for all* $S_1 \subseteq E_1$, ..., $S_l \subseteq E_l$, $R \subseteq D$.

Theorem 4. *Let* U *be a computable operation, and let* R *be equal to* $\mathrm{U}(S_1, \dots, S_l)$*. If* $S_i \in P_n$ $(n = 1, 2, \dots; i = 1, \dots, l)$*, then* $R \in P_n$*. In particular,* R *is enumerable if all* S_i *are enumerable.*

5. Kolmogorov operations.

Consider a minimal set that satisfies (a)–(c) from Section 2 and contains variables $x_1, \dots, x_n$; we denote this set by $\mathcal{H}[x_1, \dots, x_n]$. We may replace variables $x_1, \dots, x_n$ by some elements $\mathrm{a}_1, \dots, \mathrm{a}_n$ of $\mathcal{H}$ in an element $\mathrm{g}(x_1, \dots, x_n)$ of $\mathcal{H}[x_1, \dots, x_n]$. In this way, we get some element $\mathrm{g}(\mathrm{a}_1, \dots, \mathrm{a}_n)$ of $\mathcal{H}$. A Kolmogorov-type rule is a string

$$g_1(x_1, \dots, x_n), \nu_1; \quad g_2(x_1, \dots, x_n), \nu_2; \quad \dots; \quad g_m(x_1, \dots, x_n), \nu_m; \quad g(x_1, \dots, x_n), \nu. \quad (*)$$

Here, all $g_i(x_1, \dots, x_n)$ and $g(x_1, \dots, x_n)$ are elements of $\mathcal{H}[x_1, \dots, x_n]$, and m, n, ν_i, ν are natural numbers. A tuple of sets $M_1, \dots, M_k$ is closed with respect to $(*)$ if $\mathrm{g}(\mathrm{a}_1, \dots, \mathrm{a}_n) \in M_\nu$ for all $\mathrm{a}_1, \dots, \mathrm{a}_n \in \mathcal{H}$ such that $g_i(\mathrm{a}_1, \dots, \mathrm{a}_n) \in M_{\nu_i}$ for all $i = 1, \dots, m$. An l-ary Kolmogorov operation K corresponds to a finite set $\mathcal{R}$ of Kolmogorov-type rules and a tuple $k, \varkappa_1, \dots, \varkappa_l, \varkappa$ of natural numbers [where all $\varkappa_i$ and $\varkappa$ are between 1 and k]. To apply K to sets $S_1, \dots, S_l$, we do the following: Consider all tuples of sets $M_1, \dots, M_k$ that are closed under all rules in $\mathcal{R}$ such that $M_{\varkappa_i} \supseteq S_i$ for all $i = 1, \dots, l$. Then, among all these tuples, we select the tuple $M_1^+, \dots, M_k^+$ such that $M_i \supseteq M_i^+$ (for all $i = 1, \dots, k$) for every tuple in the class considered. Finally, we let $\mathrm{K}(S_1, \dots, S_l) = M_\varkappa^+$.

Theorem 5. *A mapping of* $\mathcal{V}^l$ *to* $\mathcal{V}$ *is a computable operation if and only if it is a Kolmogorov operation.*

6. Post operations.

The following definition of Post operations is a generalization and formalization of Post's ideas [4, 5]. Let A be an alphabet. Consider a set $\mathcal{S}_A[x_1, \dots, x_n]$ that is minimal among the sets having the following properties:

(a) it contains the set $\mathcal{S}_A$ of all words in the alphabet A [7];
(b) it contains variables $x_1, \dots, x_n$; and
(c) for any elements a and b, the set also contains ab.

[In other words, $\mathcal{S}_A[x_1, \dots, x_n]$ is the set of all words over the alphabet $A \cup \{x_1, \dots, x_n\}$.] A substitution, where every x_i is replaced by some word a_1 (over A), transforms an element $\mathrm{g}(x_1, \dots, x_n)$ of $\mathcal{S}_A[x_1, \dots, x_n]$ into a word $\mathrm{g}(\mathrm{a}_1, \dots, \mathrm{a}_n)$ over A. Now we take the definition of Kolmogorov's operation above and replace every occurrence of "Kolmogorov operation" by "Post operation", every occurrence of "Kolmogorov-type rule" by "Post-type rule", and every occurrence of the letter $\mathcal{H}$ by the letter $\mathcal{S}_A$. Then we get the definition of Post operation in the alphabet A. If B is some alphabet that contains A, then the Post operations in B are called Post operations *over* A.

A bijective correspondence between some subset of $\mathcal{H}$ and the set $\mathcal{S}_A$ is called a (bijective) numbering of $\mathcal{S}_A$; if an element $\mathrm{h} \in \mathcal{H}$ corresponds to an element $\mathrm{f} \in \mathcal{S}_A$, we say

that h is the number of f. A numbering is admissible if there exists a computable function $\gamma(x, y)$ (i.e., a computable partial mapping of $\mathcal{H}^2$ to $\mathcal{H}$) such that $\gamma(\mathrm{m}, \mathrm{n})$ is a number of ab whenever m and n are numbers of a and b. In particular, the numbering considered in [8] is admissible.

Theorem 6. *Consider an arbitrary admissible numbering. For a set $M \subseteq \mathcal{S}_A$, we denote by πM the set of all numbers of elements of M. An l-ary operation that maps arbitrary sets $S_1, \dots, S_l \subseteq \mathcal{S}_A$ into a set $\mathrm{P}(S_1, \dots, S_l) \subseteq \mathcal{S}_A$ is a Post operation over A if and only if there exists a computable operation* U *such that $\pi\mathrm{P}(S_1, \dots, S_l) = \mathrm{U}(\pi S_1, \dots, \pi S_l)$.*

7. Reducibility.

We say that a S is enumeration reducible to sets $S_1, \dots, S_l$ if there exists a computable operation U such that $R = \mathrm{U}(S_1, \dots, S_l)$. If S_i and R are subsets of $\mathbb{N}$, then (according to Theorem 3), we may assume[2] that U is a mapping of $\mathbb{N}^l$ to $\mathbb{N}$.

Theorem 7. *Consider an operator* F *that can be applied to any* [*total*] *functions $\psi_1, \dots, \psi_l$ with $m_1, \dots, m_l$ natural number arguments and natural number values and that produces an n-ary function $\varphi = \mathrm{F}(\psi_1, \dots, \psi_l)$ with natural number arguments and values. Then the operator* F *is partial recursive if and only if there exists a computable operation* U *such that $G_{\mathrm{F}(\psi_1,\dots,\psi_l)} = \mathrm{U}(G_{\psi_1}, \dots, G_{\psi_l})$.*

(Using Theorem 3, we may assume[3] that U is a mapping of $\mathbb{N}^{m_1+1} \times \dots \times \mathbb{N}^{m_l+1}$ to $\mathbb{N}^{n+1}$.)

Corollary. *A function φ is computably reducible to* [*total*] *functions $\psi_1, \dots, \psi_l$ if and only if its graph is enumeration reducible to the graphs of $\psi_1, \dots, \psi_l$.*

Theorem 8. *A set P is* (*decidably*) *reducible to a set Q if and only if each of the sets P and $\overline{P}$ is enumeration reducible to the sets Q and $\overline{Q}$.* (Here, $\overline{P}$ and $\overline{Q}$ are complements to P and Q; cf. [5].)

Corollary. *Assume that P and Q are enumerable. Then P is* (*decidably*) *reducible to Q if and only if $\overline{P}$ is enumeration reducible to $\overline{Q}$.*

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References

[1] Hilbert, D., *Grundlagen der Geometrie*, Moscow, 1948. Гильберт Д. *Основания геометрии*, Москва: ОГИЗ, 1948, добавление VII.

[2] Kleene, S.C. *Introduction to Metamathematics*, North-Holland Pub. Co., 1952.

[3] Kleene, S.C.; Post, E.L., The upper semi-lattice of degrees of recursive unsolvability. *Annals of Mathematics*, vol. 59 (1954), no. 3, pp. 379–407.

[2]Seems to be an error: $\mathbb{N}$ should be replaced by the set of all subsets of $\mathbb{N}$.

[3]Seems to be an error: functions are subsets of $\mathbb{N}^{m_i+1}$, not elements.

[4] Post, E.L. Formal reductions of the general combinatorial decision problem. *American Journal of Mathematics*, vol. 65 (1943), num. 2, pp. 197–215. DOI: `10.2307/2371809`.

[5] Post, E.L. Degrees of recursive unsolvability, abstract (preliminary report). *Bulletin of the American Mathematical Society*, vol. 54 (1948), no. 7, p. 641–642.

[6] Mostowski, A. On definable sets of positive integers. *Fundamenta Mathematicae*, vol. 34 (1947), num. 1, pp. 81–112.

[7] Markov, A.A., *Theory of Algorithms*, Trudy Mat. Inst. Steklov Akad. SSSR, vol. 24, 1954, 376 pages. Марков А.А. *Теория алгорифмов*. Труды Математического института им. Стеклова АН СССР, том 42, М.–Л.: Изд-во АН СССР, 1954, 376 с.

[8] Uspensky, V.A., *Gödel's Theorem and the theory of algorithms*, Doklady Akad. SSSR, vol. 91 (1953), issue 4, pp. 737-740. Успенский В.А. Теорема Гёделя и теория алгоритмов. *Доклады АН СССР*, т. 91 (1953), № 4, с. 737–740.

Translator's note. This paper published in 1955 seemed to be available only in Russian and probably was not noticed elsewhere. Still, it contains the first published definition of enumeration reducibility (an equivalent definition was published by Rogers and Friedberg in 1959, see the discussion in `https://arxiv.org/pdf/2008.02773.pdf`).

The definition uses cumbersome notation; here is the reformulation. Consider the set $\mathcal{P}(\mathbb{N})$ of all sets of natural numbers as a topological space. Basic open sets correspond to finite sets $X \subset \mathbb{N}$; for each finite set X, we consider the family B_X of all its supersets, and declare all B_X and their unions as open sets. Now we consider continuous mappings $F : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$. They can be equivalently defined by two properties: (a) F is monotone, i.e., $F(X) \subset F(Y)$ if $X \subset Y$; (b) if $F(X)$ contains some n, then there exists a finite subset $X' \subset X$ such that $F(X')$ contains this n.

Now we say that a mapping $F : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ is computable if F is continuous and the set of pairs (X, x) such that X is a finite set and $x \in F(X)$ is enumerable. If $F(Y) = X$ for some computable mapping F, we say that X is enumeration reducible to Y. Similarly, we define the notion of enumeration reducibility of X to $Y_1, \ldots, Y_n$ using computable continuous mappings of type $(\mathcal{P}(\mathbb{N}))^n \to \mathcal{P}(\mathbb{N})$.

In addition to this definition, the paper contains (Sections 5 and 6) equivalent definitions based on specific computation (enumeration) models. Section 6 suggests considering Post production systems (introduced in [4] as a tool for generating enumerable sets) in an extended version, where elements of some other set Y are added as "axioms". Then the generated set depends on Y, this dependence is a computable operation, and all computable operations can be obtained in this way. Section 5 suggests another formalism going back to Uspensky's advisor Kolmogorov. (There are no publications by Kolmogorov that mention this approach.) The main difference with Post's approach is that lists (composed of a basic symbol |) are considered instead of strings, and derivation rules are formulated in terms of list rewriting defined by lists with free variables.

These two approaches are claimed to be equivalent to the main definition. The proof is not given in the paper; it appeared in the PhD thesis of Uspensky `https://archive.org/details/uspensky_thesis_1955/` (in Russian). Still, it is not surprising: The Post production calculus is universal, and it is natural to expect that it remains universal (for the same reasons) if we allow additional axioms. Kolmogorov's approach looks more exotic, though.

(Translated by Alexander Shen using the Russian TEX version prepared by Evgeny Zolin; we thank our colleagues for editing the translation. Footnotes and commentaries in square brackets were added by the translator.)